CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Modeling Situations with Linear Equations

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Modeling Situations with Linear Equations

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students use algebra in context, and in particular, how well students:

- Explore relationships between variables in everyday situations.
- Find unknown values from known values.
- Find relationships between pairs of unknowns, and express these as tables and graphs.
- Find general relationships between several variables, and express these in different ways by rearranging formulae.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

8.F Construct a function to model a linear relationship between two quantities.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

2. Reason abstractly and quantitatively.
4. Model with mathematics.

INTRODUCTION

This activity links several aspects of algebra. A situation is presented, and relationships between the variables are explored in depth. Letters are used to represent unknowns, generalized numbers, and variables. The problem-solving context gives you a chance to assess how well students are able to combine and apply different aspects of their algebra knowledge and skills.

- Before the lesson, students work individually on the assessment task *The Guitar Class*. You then review their work and create questions for students to answer at the end of the lesson, to help them to improve their solutions.
- During the lesson, students translate between words, algebraic formulae, tables, and graphs in an interactive whole-class discussion. The intention is that you focus students on making sense of the context using algebra, rather than just the routine use of techniques and skills. Students then work in pairs to graph the relationship between two of the variables. In a final whole-class discussion, students identify general formulae showing the relationships between all the variables.
- At the end of the lesson, students review and improve their individual work.

MATERIALS REQUIRED

- Each student will need two copies of the assessment task *The Guitar Class*, either the lesson task sheet *Making and Selling Candles,* or the lesson task sheet *Rescue Helicopter*, depending on the outcome of the assessment task, a mini-whiteboard, a pen, and an eraser.
- Each pair of students will need a sheet of graph paper.
- There are some projector resources to support whole-class discussion.

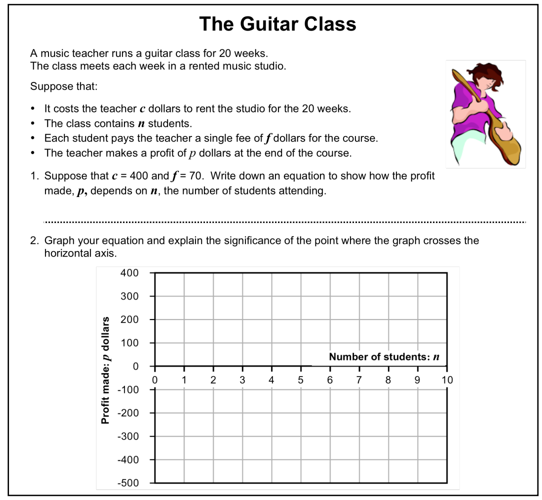
TIME NEEDED

15 minutes before the lesson and a one-hour lesson. All timings are approximate. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *The Guitar Class* (15 minutes)

Have the students do this task in class or for homework, a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. Then you will be able to target your help more effectively in the follow-up lesson.


The Guitar Class

A music teacher runs a guitar class for 20 weeks.
The class meets each week in a rented music studio.

Suppose that:

- It costs the teacher c dollars to rent the studio for the 20 weeks.
- The class contains n students.
- Each student pays the teacher a single fee of f dollars for the course.
- The teacher makes a profit of p dollars at the end of the course.

1. Suppose that $c = 400$ and $f = 70$. Write down an equation to show how the profit made, p, depends on n, the number of students attending.

....

2. Graph your equation and explain the significance of the point where the graph crosses the horizontal axis.

Give out *The Guitar Class*. Introduce the task briefly, and help the class to understand the problem and its context.

> *Read through the questions, and try to answer them as carefully as you can.*
>
> *What does 'profit' mean?*
>
> *Don't worry too much if you can't understand and do everything.*
>
> *I will teach a lesson with a task like this [tomorrow].*
>
> *By the end of that lesson, your goal is to answer the questions with confidence.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding. The purpose of this is to forewarn you of issues that will arise during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' work. Research suggests this will be counterproductive, as it encourages students to compare their scores and distracts their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest that you write a list of your own questions, based on your students' work, using the ideas that follow. You may choose to write questions on each student's work. If you do not have time to do this, just select a few questions that will be of help to the majority of students. These can be written on the board at the end of the lesson.

Choosing the lesson task

After writing your list of questions, use your assessment of students' current understanding to decide which task to use during the lesson. We have found that many students learn from the *Making and Selling Candles* task. However, if the majority of your students have answered most of the assessment task questions correctly, set the *Rescue Helicopter* task instead. *Making and Selling Candles* is a more structured task than *Rescue Helicopter*, so makes less demand on students' problem solving skills.

Common issues:	**Suggested questions and prompts:**
Student writes no equation, or just a few numbers and letters (Q1)	• What do you know from the question? • Suppose you have 10 students. How do you figure out how much profit the teacher would make?
Student writes an equation with a particular value of *n* (Q1) For example: The student substitutes $n = 30$.	• Does the calculation method change as you vary n? • How can you write the calculation for any value of n?
Student uses incorrect operation in equation (Q1) For example: The student divides the cost by the number of students, and adds rather than subtracts c to get $p = 70n + c$.	• How much money does each student pay? How much money do the students pay altogether? • Is the amount of money you have calculated before paying costs more or less than the profit?
Student draws incorrect graph (Q2) For example: The graph has negative slope, or is not a straight line.	• What do you think happens to the amount of profit as the number of students increases? • What kind of function links n and p? • What graph would you expect from this equation?
Student does not explain or misinterprets the significance of the x-intercept (Q2) For example: The student does not link the answer back to the context. She just writes $p = 5.5$, and does not mention that this is the point at which the teacher begins to make a profit.	• What does n stand for? Does your answer make sense? • Reread the first part of the sheet. What does $p > 0$ mean?
Student uses incorrect operations in formulas in Q3, Q4 For example: The student writes $p = fn + c$ or $f = pn - c$.	• What does 'profit' mean? • What does f stand for? • Explain how you would calculate the profit in words.
Student answers Q1-5 correctly	• Suppose you can make 100 candles from a kit costing \$70. Use the profit you would expect to make to calculate the selling price. Graph this relationship.

SUGGESTED LESSON OUTLINE

Interactive whole-class introduction (30 minutes)

Give the class copies of the situation *Making and Selling Candles* and the mini-whiteboards.

Work through this sheet with the class, using episodes of whole-class discussion interspersed with short episodes of paired or individual work. Keep all students interested by asking them to show their answers using the mini-whiteboards. When a student offers an answer, ask other students to comment or explain, rather than evaluating the answer yourself.

Students' spoken and whiteboard responses will give you information about what they are finding difficult. Get students to work together in small groups on those parts of the task so as to produce a joint response. Use the joint responses as the basis for further class discussion.

Questions 1 and 2

For example, you could work on Questions 1 and 2 orally.

What numbers should I write in for k, n and s [Question 1]?

How could we figure out the total profit made, p, from the other numbers [Question 2]?

Use your mini-whiteboards to show me the equation you need to solve.

Allow students a few minutes to discuss their ideas in pairs and then ask them to present their equations. Ask students to justify every step. Keep linking the context and the representation. For example:

You have to multiply 4 by 60 [or n by s].

What does that tell you? [The amount of money he makes from selling the candles.]

What do you have to do next? [Subtract 50 (or k).]

Why subtract? [Because the cost of buying the kit will reduce his profit.]

How can I write this equation using the values of k, n, s?

Students will probably find more than one way of writing the equation.

When asking students to show equivalence, ask them to explain what the equation means in words, not just rearrange the equation.

How else can I write this equation using the values of k, n, s?

Anthony, you wrote the equation a different way. How did you write it?

Are these two equations the same? How do you know?

Kay, do you agree?

Ask students to explore different values of one of the variables:

What if you charged \$3 for each candle? \$7?

What other values could n take?

What else could I change?

What method do I use to calculate the value of p?

Does the calculation method change when I change the value of n? s? k?

In this way, show that $p = 60 \times 4 - 50$ and that in general, $p = ns - k$.

This generality is not a trivial step. By calculating using different values for each variable, students should become aware that the **same method** works whatever values are given to those variables. You may need to spend some time establishing this point.

Give students two or three minutes to record their solutions to Question 1 and Question 2, then ask them to read Question 3.

Question 3

Now look together at Question 3. Insert the answer for p as 190, and erase the selling price, s.

> *Suppose we know the cost of the kit, the number of candles that may be made from the kit, and the profit we want to make. How could we figure out the correct selling price?*
>
> *Write your method on your mini-whiteboard.*
>
> *I see $p + k$ and $p - k$. Which do you think is correct? Why? [You first add the cost of buying the kit to the profit, 50 + 190 or $p + k$.]*
>
> *What does that tell you? [The total amount of money you need to make from selling the candles.]*
>
> *Then what do you do? [You divide the answer by the number of candles to find the selling price per candle.]*

Students should reason that $s = \dfrac{190 + 50}{60}$ and that $s = \dfrac{p + k}{n}$.

To focus all students on understanding the relationship between the context and the algebra, it can be useful to get a number of different students to explain the same answer.

> *Alicia – please explain why Alex is correct, in your own words.*
>
> *Kay – you explain it too, in your own words, please.*

Ask students to explain whether the same method works for different values.

> *What method do you use to calculate the value of s?*
>
> *Does the calculation method change when I change the value of n? p? k?*

If students produce different equations, get them to link their descriptions of the situation with the different numerical and algebraic representations:

> *Are these two ways of writing the same numeric equation? How do you know?*
>
> *Are these two ways of writing the algebraic equations equivalent? How do you know?*

Finally, focus students' attention on what they have been doing when they know three values and solve to find the fourth:

> *You've figured out how to solve for p, and how to solve for s. What else could I change?*

Question 4

Erase *two* numbers, n and p:

		k	
The cost of buying the kit: (This includes the molds, wax and wicks.)	$		
		n	
The number of candles that can be made with the kit:			candles
		s	
The price at which he sells each candle:	$		per candle
		p	
Total profit made if all candles are sold:	$		

What numbers could n and p be?
Is this the only solution? Give me another solution… and another… and another…

This time there are two numbers that I don't know, n and p. What is different from not knowing one number?

Write a table of possibilities and draw me a graph to show how p depends on n.

Working in pairs (10 minutes)

Hand out graph paper and allow time for students to work on the problem in pairs. As they do this, go round and prompt them to try different pairs of numbers, using the structure of the situation:

If n were 20, what would this equation mean?

How could you calculate the profit?

How can you write this method in words or symbols?

n	20	30	40	50
p	30	70	110	150

What equation fits this table?

What would the graph look like?

What does the graph show?

During the paired work you have two tasks: to note student difficulties and to support them in their thinking about the graphing activity.

Note student difficulties

Students have not been given a table to complete, nor is there a set of axes or scales on the axes. This allows you to find out whether students are able to use their knowledge in context.

- Is students' use of algebraic notation accurate? Conventional?
- Do they choose a sensible range for the number of students when drawing the graph, including values of n from 0 to 60?
- Do students calculate an appropriate scale for each axis to fit all sensible values of n and p?

- Do they label axes accurately and use equal increments?
- Do they know how to draw and complete an accurate table of values?
- Do they plot the points accurately?
- Do they make a point plot, or join the points to make a linear graph?
- Do students use features such as intercepts and linearity to check the accuracy of their plots?

Support students' thinking

If students are stuck or making errors, try to support their thinking rather than solve the problems for them. If a pair of students is stuck on a problem, suggest that they seek help from another pair who have dealt with that problem successfully.

You have chosen values of n from x to y. Why did you choose those values?

How do you know that your plot is accurate?

What does the x-intercept show?

Encourage students to continue to link their mathematical representations with the context.

What happens to the profit if the kit only makes five candles?

At this point, $n = 3.5$. Does that make sense?

If students complete this quickly, set one of the following questions:

Is the graph best drawn as a point graph or as a linear graph? Explain your answer.

Explain the relevance of the x- and y-intercepts.

Erase two different variables [s and p, or n and s] and express the relationship between them if all other variables are kept constant.

Whole-class discussion (10 minutes)

Ask students to try to answer Question 5 in pairs. Although students have already done part of this question, they need to recognize this for themselves.

Q5: Write a general formula for showing the relationships between the variables. What do you think you are being asked to do?

Suppose you know s, k, and n and you want to calculate p. What method are you going to use?

Does the method change if you change the value of n?

What's the method for calculating p, whatever the values of s, k, and n?

If students are really stuck, refer them back to their earlier work:

Look back at Q2. What did you do to calculate s when you knew p, k, n?

What would happen to the calculation method if n were any other value?

Tell me in words how to calculate the total amount of money you collect. How would you write that using algebra? How do you calculate the selling price from that?

After a few minutes, ask each pair to show you the four general formulas using their mini-whiteboards:

$$p = ns - k \qquad k = ns - p \qquad n = \frac{p+k}{s} \qquad s = \frac{p+k}{n}$$

If there is disagreement about formulas, write different versions on the board. Ask students to show the equivalence of different formulas.

Ask students to say how they figured out their answers. Hopefully, some will have focused on writing relationships from the situation, generalizing numerical versions of the equation, and others will have used algebraic manipulation.

Improve individual solutions to the assessment task (10 minutes)

If you are running out of time, you could set this task in the next lesson or for homework.

Give students back their work on assessment task *The Guitar Class*, along with a fresh copy of that task sheet.

Work on your own for ten minutes.

[Remember the work you did on The Guitar Class.]

I'm giving you your own answers, and a new sheet to work on.

Read through your original solution and think about what you learned last lesson.

I want you to use what you learned to improve your solution to The Guitar Class. Then compare your answers to see what progress you made.

SOLUTIONS

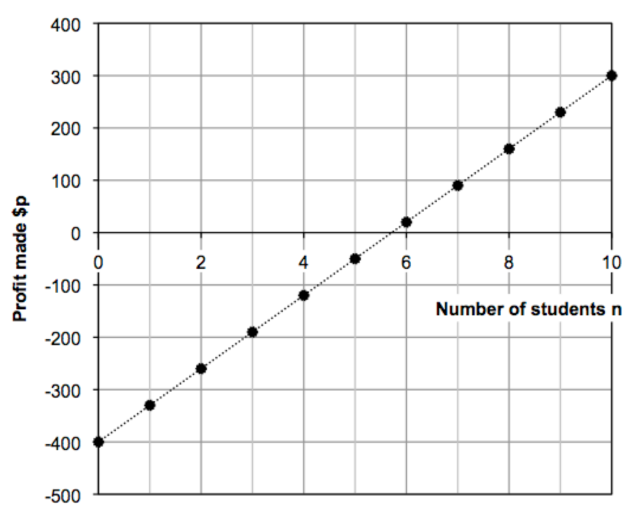
The Guitar Class

1. $p = 70n - 400$

2. The point where the graph crosses the horizontal axis is helpful in identifying the number of students that must attend before the music teacher will break even. Thus 5 students will make a loss, but 6 students will make a profit.

3. $p = nf - c$

4. $f = \dfrac{p + c}{n}$



Making Candles

2. The student may give a description such as "You find how much money is made by selling them then take off the cost of the kit", a formula such as $p = ns - k$, or a written calculation such as $p = 60 \times 4 - 50$.

   The calculation method stays the same: you always subtract the costs from the revenue to find profit.

3. For example: $s = \dfrac{p+k}{n}$ or $s = \dfrac{190+50}{60}$.

   As in Question 2, the method stays the same whatever values are substituted.

4. Look for a table of values with an appropriate range of values for n, including cases when p is negative. Check that the graph shows values in equal increments along the axes, is plotted accurately, shows the intercepts, and that the points fit a straight line. The x-intercept is (12.5,0): since you cannot sell half a candle (n is a discrete variable) this means that at least 13 candles must be sold in order to make a profit. You could discuss whether or not it is correct to draw a line joining the points.

5. $p = ns - k$ $\quad k = ns - p$ $\quad s = \dfrac{p+k}{n}$ $\quad n = \dfrac{p+k}{s}$.

Rescue Helicopter

The formulas are: $t = \dfrac{d}{s} + w$ $\quad d = s(t-w)$ $\quad s = \dfrac{d}{t-w}$ $\quad w = t - \dfrac{d}{s}$.

Stress that t is the total duration, not the time taken to cover the distance.

The Guitar Class

A music teacher runs a guitar class for 20 weeks.
The class meets each week in a rented music studio.

Suppose that:

- It costs the teacher c dollars to rent the studio for the 20 weeks.
- The class contains n students.
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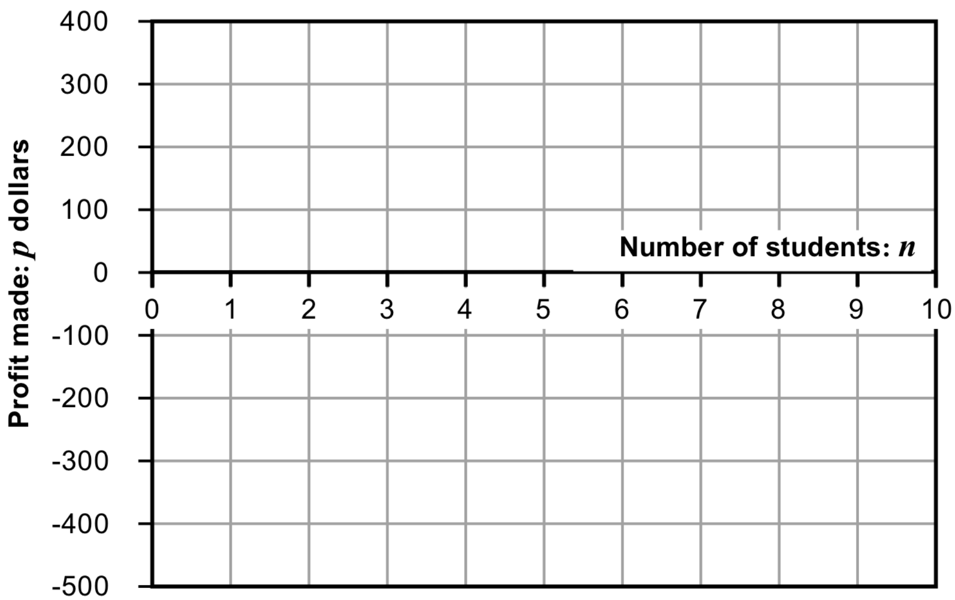
1. Suppose that $c = 400$ and $f = 70$. Write an equation to show how the profit made, p, depends on n, the number of students attending.

2. Graph your equation and explain the significance of the point where the graph crosses the horizontal axis.



3. Write a formula for calculating p when you know c, n, and f.

4. Write a formula for calculating f when you know c, n, and p.

Making and Selling Candles

A student wants to earn some money by making and selling candles.

Suppose that he can make 60 candles from a $50 kit, and that these will each be sold for $4.

		k		
The cost of buying the kit: (This includes the molds, wax and wicks.)	$			
		n		
The number of candles that can be made with the kit:			candles	
		s		
The price at which he sells each candle:	$		per candle	
		p		
Total profit made if all candles are sold:	$			

1. Write the values for k, n, and s into the table above.
2. How can you calculate the profit p using the given values of k, n, and s?

   Would your method change if the values of k, n, and s were different? Explain your answer.

3. Now that you know the profit, erase the selling price of each candle, s.
   The values of k, n, and p are in the table.
   Suppose you didn't know s. How could you figure it out?

   Would your method change if the value of each variable were different? Explain your answer.

4. Now **erase two** numbers: n and p.

		k	
The cost of buying the kit: (This includes the molds, wax and wicks.)	$		
		n	
The number of candles that can be made with the kit:			candles
		s	
The price at which he sells each candle:	$		per candle
		p	
Total profit made if all candles are sold:	$		

What could these numbers be?

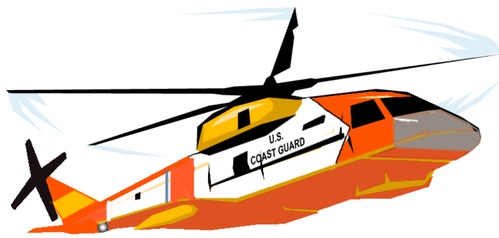
Construct a table of possible values.
Plot a graph to show the relationship.

5. Write down four general formulas showing the relationships between the variables.

p = ... s = ...

n = ... k = ...

Rescue Helicopter



	w		
Time to load and warm up the helicopter before take-off:	5	minutes	
	s		
The average speed of the helicopter in flight:	1.5	miles per minute	
	d		
The distance flown to the accident:	60	miles	
	t		
The total time needed to arrive at the scene of the accident:	45	minutes	

Hide each number in turn.

If you didn't know this number, how might it be found from the other numbers?

Hide two numbers.

What could they be? Construct a table of possible values.

Sketch a graph to show the relationship between these numbers.

Repeat this with another pair of numbers.

Hide all the numbers.

Construct a general formula for the relationship between them.

Try to write your formula in different ways, starting $t=\ldots$, $d=\ldots$, and so on.

Making and Selling Candles

			k	
The cost of buying the kit: (This includes the molds, wax and wicks.)	$			
			n	
The number of candles that can be made with the kit:				candles
			s	
The price at which he sells each candle:	$			per candle
			p	
Total profit made if all candles are sold:	$			

Rescue Helicopter

Time to load and warm up the helicopter before take-off:	*w*		minutes
The average speed of the helicopter in flight:	*s*		miles per minute
The distance flown to the accident:	*d*		miles
The total time needed to arrive at the scene of the accident:	*t*		minutes

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
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It was refined on the basis of reports from teams of observers led by
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based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
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by
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and based at the University of California, Berkeley

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